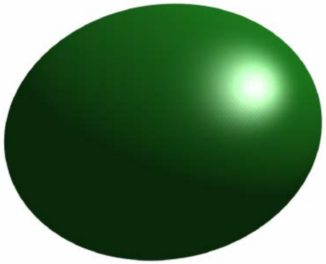
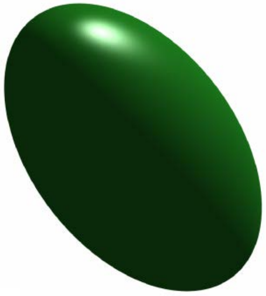


Analytical Expressions for Deformation from an Arbitrarily Oriented Spheroid in a Half-Space

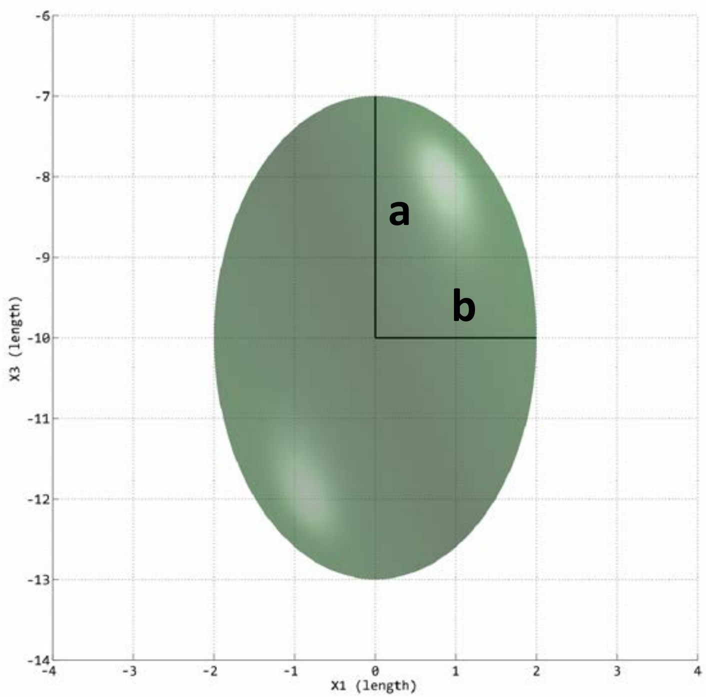
Peter F. Cervelli
USGS Volcano Science Center
December 12, 2013



U.S. Department of the Interior
U.S. Geological Survey

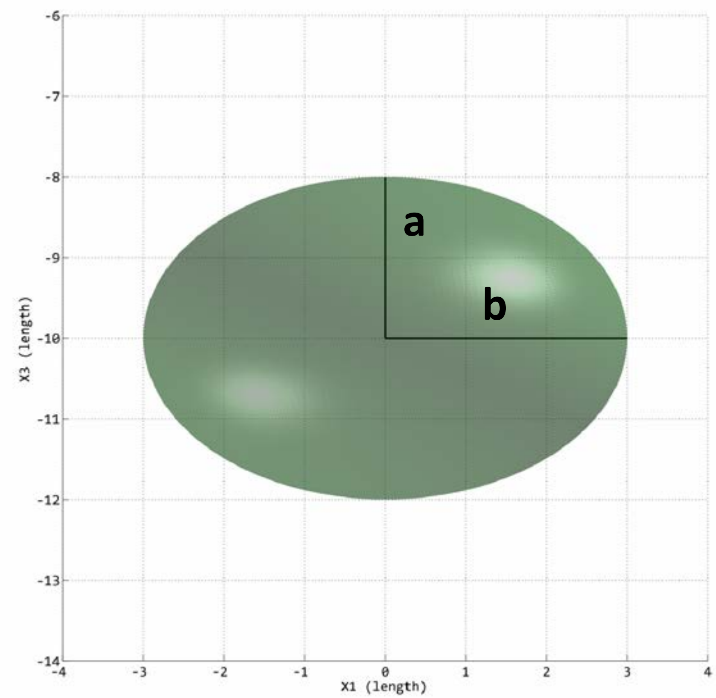
A **spheroid** is an ellipsoid having two axes (**b**) of equal length.

Prolate



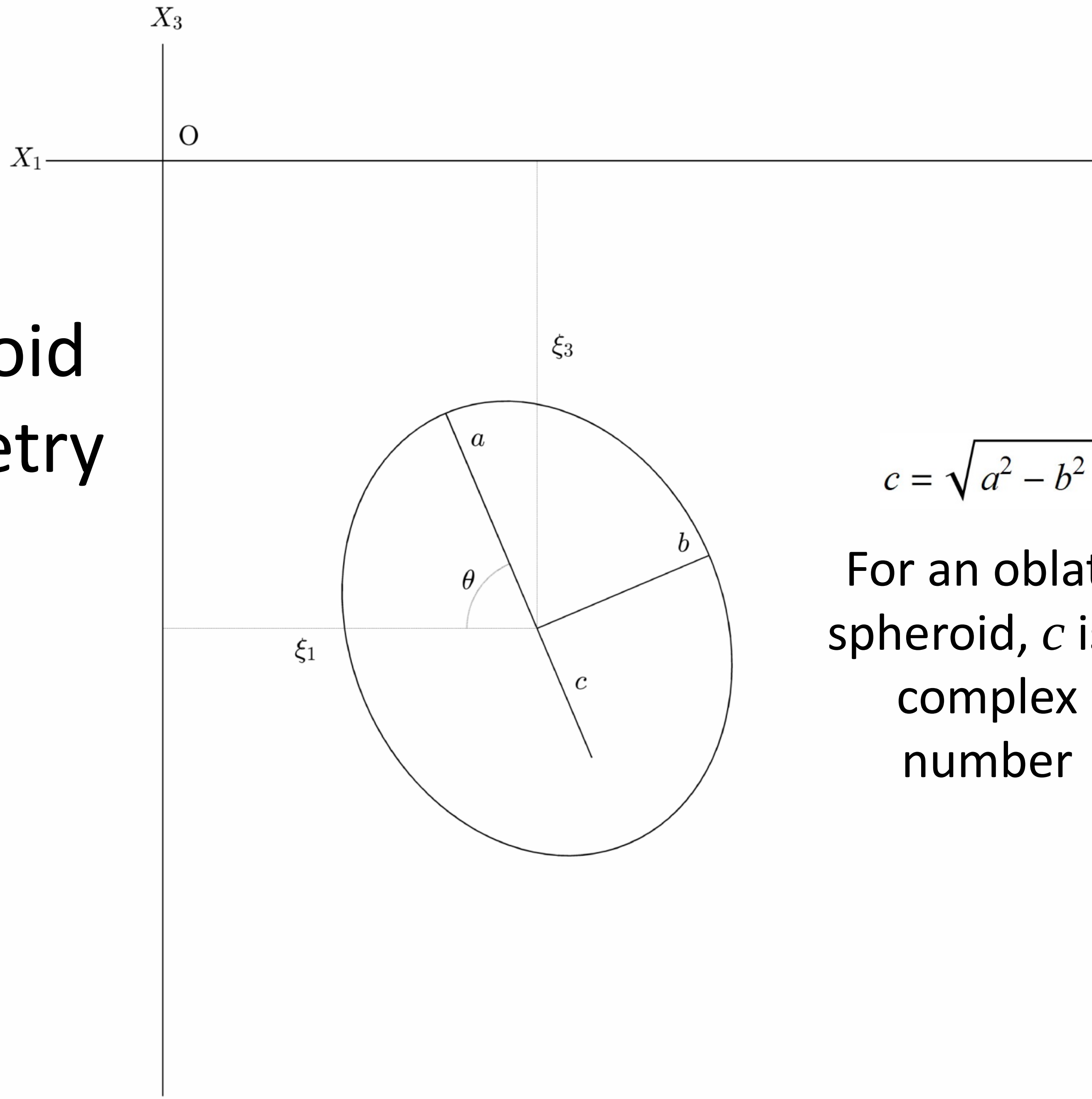
a > b

Oblate



b < a

Spheroid Geometry



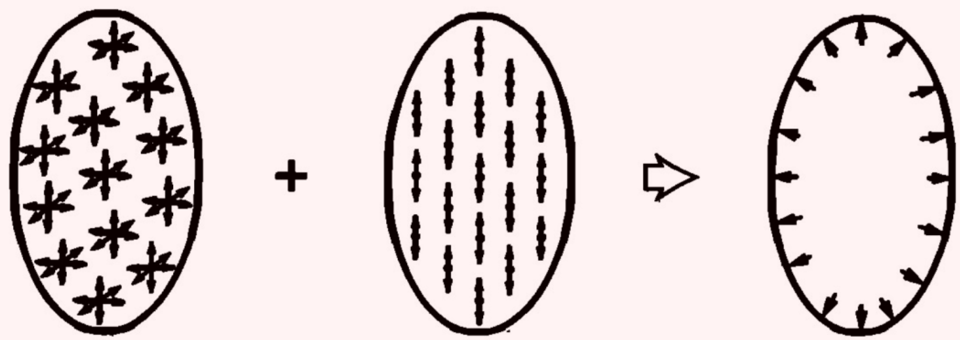
$$c = \sqrt{a^2 - b^2}$$

For an oblate spheroid, c is a complex number

Satisfying the Uniform Internal Pressure Boundary Condition

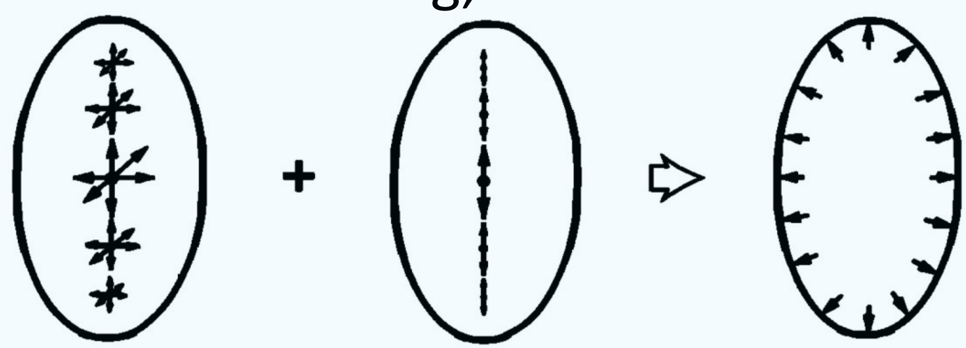
Uniform distribution,
constant strength

Eshelby, 1957



Line distribution,
variable strength

Yang, 1988



Strength varies quadratically

Centers of Dilatation

Double Forces

Uniform Internal Pressure

Volterra's Equation for Displacement as Expressed by Yang et al., 1988

Center of Dilatation	Horizontal Double Force	Vertical Double Force	Mixed Double Forces

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \left(\frac{\partial G_{i,1}}{\partial \xi_1} \cos^2(\theta) + \frac{\partial G_{i,3}}{\partial \xi_3} \sin^2(\theta) + \left(\frac{\partial G_{i,1}}{\partial \xi_3} + \frac{\partial G_{i,3}}{\partial \xi_1} \right) \sin(\theta) \cos(\theta) \right) \right) d\xi$$

$G_{i,j}$ = Point Force Green's functions

P_c = Strength function for Center of Dilatation

P_d = Strength function for Double Forces

θ = Spheroid Dip

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Volterra's Equation for Displacement as Expressed by Yang et al., 1988

Center of Dilatation ↓ | Horizontal Double Force ↓ | Vertical Double Force ↓ | Mixed Double Forces ↓

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \left(\frac{\partial G_{i,1}}{\partial \xi_1} \cos^2(\theta) + \frac{\partial G_{i,3}}{\partial \xi_3} \sin^2(\theta) + \left(\frac{\partial G_{i,1}}{\partial \xi_3} + \frac{\partial G_{i,3}}{\partial \xi_1} \right) \sin(\theta) \cos(\theta) \right) \right) d\xi$$

$G_{i,j}$ = Point Force Green's functions

$P_c = \alpha_1 \xi^2 + \alpha_2 \xi + \alpha_3$

$P_d = \beta_1 \xi^2 + \beta_2 \xi + \beta_3$

θ = Spheroid Dip

Volterra's Equation for Displacement as Expressed by Yang et al., 1988

Center of Dilatation → Horizontal Double Force → Vertical Double Force → Mixed Double Forces

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2\, \mu \left(\frac{\partial G_{i,1}}{\partial \xi_1} \cos^2(\theta) + \frac{\partial G_{i,3}}{\partial \xi_3} \sin^2(\theta) + \left(\frac{\partial G_{i,1}}{\partial \xi_3} + \frac{\partial G_{i,3}}{\partial \xi_1} \right) \sin(\theta) \cos(\theta) \right) \right) d\xi$$

$G_{i,j}$ = Point Force Green's functions

$P_c = a_1 - b_1 (\xi^2 + c^2)$

$P_d = a_1 \nu (\xi^2 + c^2)/b^2$

θ = Spheroid Dip

a_1 and b_1 are the unknown coefficients that define the strength functions

Methodology of Solution

1. Solve integral using full space Green's functions, assuming $\theta = 90°$.

2. Determine parameters (a_1 and b_1) defining strength functions by using uniform pressure boundary condition.

3. Solve integral using half space Green's functions and combine with strength function parameters found above.

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1. Solve Integral with Full Space Green's Functions

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \left(\frac{\partial G_{i,1}}{\partial \xi_1} \cos^2(\theta) + \frac{\partial G_{i,3}}{\partial \xi_3} \sin^2(\theta) + \left(\frac{\partial G_{i,1}}{\partial \xi_3} + \frac{\partial G_{i,3}}{\partial \xi_1} \right) \sin(\theta) \cos(\theta) \right) \right) d\xi$$

$\theta = 90°$

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \, \frac{\partial G_{i,3}}{\partial \xi_3} \right) d\xi$$

Displacements From a Spheroid in a Full Space

$$\text{U1} = \frac{\text{P}}{4\ \text{mu}} \text{x1} \left(\text{a1} \left(\frac{\text{b}^2}{\text{R}\ (\text{R} - \text{x3} - \xi)} - \frac{\xi}{\text{R}} - \text{Log}[\text{R} - \text{x3} - \xi] \right) - \text{b}^2\ \text{b1} \left(\frac{\text{x3} - \xi}{\text{R} - \text{x3} - \xi} - \text{Log}[\text{R} - \text{x3} - \xi] \right) \right);$$

$$\text{U2} = \text{U1} \frac{\text{x2}}{\text{x1}};$$

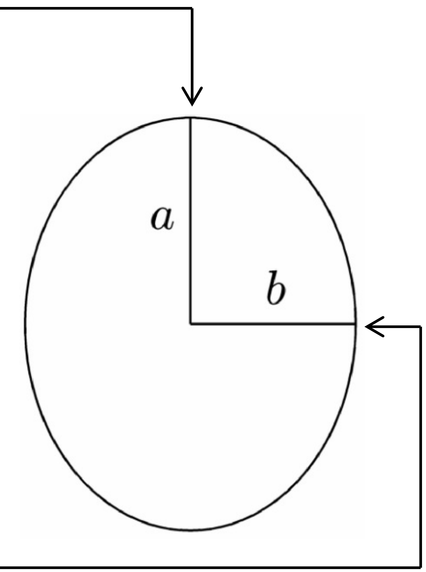
$$\text{U3} = \frac{\text{P}}{4\ \text{mu}} \left(\text{a1} \left(\text{R} - \frac{\text{b}^2 + \xi\ (\text{x3} + \xi)}{\text{R}} + 4\ (1 - \text{nu})\ (\text{R} + \text{x3}\ \text{Log}[\text{R} - \text{x3} - \xi]) \right) - 2\ \text{b}^2\ \text{b1}\ (\text{R} + \text{x3}\ \text{Log}[\text{R} - \text{x3} - \xi]) \right);$$

$$\text{R} = \sqrt{\text{x1}^2 + \text{x2}^2 + (\text{x3} + \xi)^2};$$

Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2012).

2. Determine strength function parameters

```
Solve[
  {P (a1 ((-3 + 2 nu) c/a + (-1 + 2 nu)/2 Log[(a - c)/(a + c)]) + b1/2 (2 c (b^2 - c^2)/a + b^2 Log[(a - c)/(a + c)])) == P,
   -P (a1 ((-1 + nu) (2 a c)/b^2 + (-2 + nu) Log[(a - c)/(a + c)]) + b1 (2 a c + b^2 Log[(a - c)/(a + c)])) == P
  },
  {a1, b1}
] // Simplify
```



$$\left\{\left\{a1 \to \frac{b^2\left(2c\left(2a^2+b^2-c^2\right)+3ab^2\,\mathrm{Log}\left[\frac{a-c}{a+c}\right]\right)}{4ac^2\left(b^2(-2+nu)+c^2(-1+nu)\right)+2b^2c\left(c^2(-2+nu)+b^2(-1+nu)+a^2\,nu\right)\mathrm{Log}\left[\frac{a-c}{a+c}\right]+ab^4(1+nu)\,\mathrm{Log}\left[\frac{a-c}{a+c}\right]^2},\right.\right.$$

$$\left.\left.b1 \to \frac{2c\left(b^2(3-2nu)-2a^2(-1+nu)\right)+ab^2(5-4nu)\,\mathrm{Log}\left[\frac{a-c}{a+c}\right]}{4ac^2\left(b^2(-2+nu)+c^2(-1+nu)\right)+2b^2c\left(c^2(-2+nu)+b^2(-1+nu)+a^2\,nu\right)\mathrm{Log}\left[\frac{a-c}{a+c}\right]+ab^4(1+nu)\,\mathrm{Log}\left[\frac{a-c}{a+c}\right]^2}\right\}\right\}$$

Values for a_1 and b_1 depend only on geometry of spheroid and elastic constants

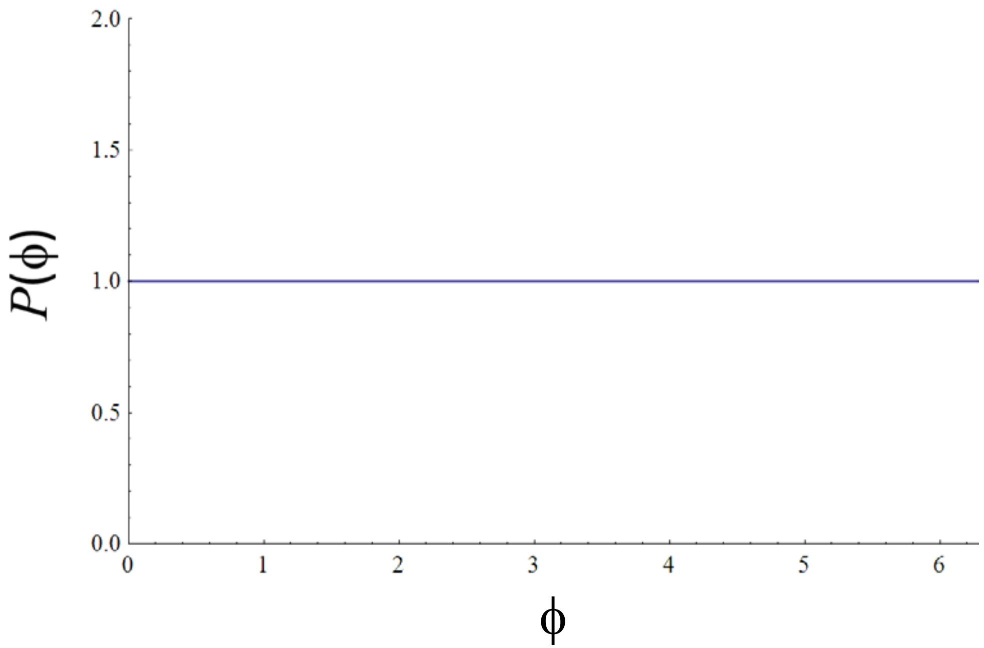
2. (continued) Check Boundary Condition

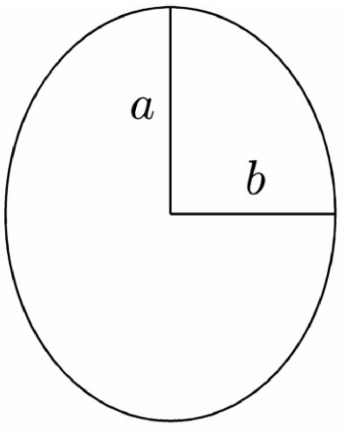
Plugging in the values for a_1 and b_1 derived above, we can test the uniform pressure boundary condition

$$P(\phi) = \mathbf{n}^{\mathrm{T}}\mathbf{S}\,\mathbf{n}$$

Stress tensor as function of ϕ

Surface normal vector as function of ϕ





3. Solve Integral with Half Space Green's Functions

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \left(\frac{\partial G_{i,1}}{\partial \xi_1} \cos^2(\theta) + \frac{\partial G_{i,3}}{\partial \xi_3} \sin^2(\theta) + \left(\frac{\partial G_{i,1}}{\partial \xi_3} + \frac{\partial G_{i,3}}{\partial \xi_1} \right) \sin(\theta) \cos(\theta) \right) \right) d\xi$$

JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 93, NO. B5, PAGES 4249–4257, MAY 10, 1988

Deformation From Inflation of a Dipping Finite Prolate Spheroid in an Elastic Half-Space as a Model for Volcanic Stressing

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Department of Earth and Space Sciences, University of California, Los Angeles, California

JAMES H. DIETERICH
U.S. Geological Survey, Menlo Park, California

Although Yang's 1988 paper has typos, the presented solution is otherwise correct.

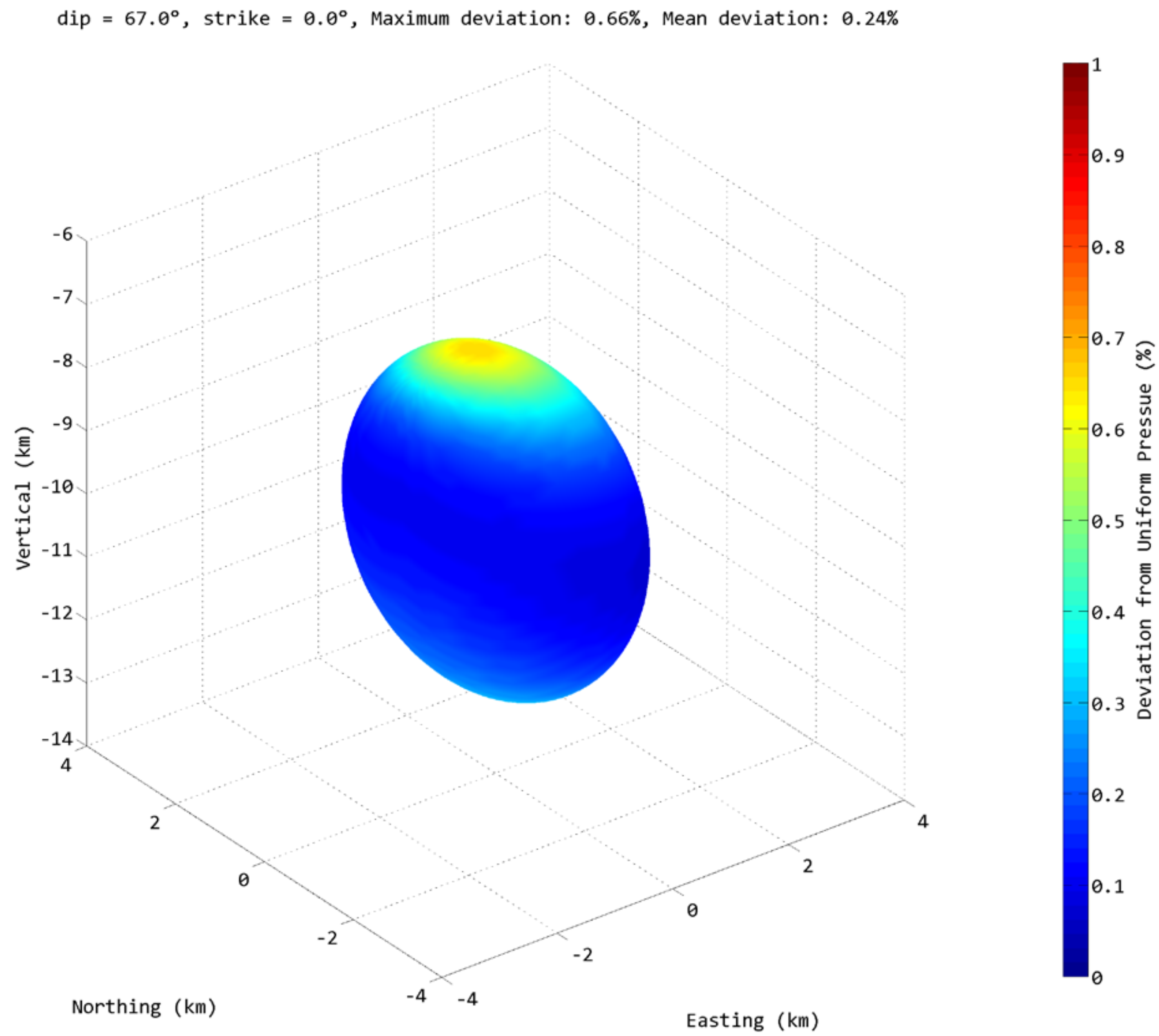
My work extends Yang's by providing:

- Displacement derivatives, strains, and stresses
- Properly handled limiting cases
- Proof that Yang's method works for oblate spheroids
- A very good approximation to the volume/pressure relationship
- Error-free code, validated by Mathematica

How Well is the Uniform Pressure Boundary Condition Satisfied?

Spheroid Parameters

a = 3000 m
b = 2000 m
Depth = 10000 m
P/μ = 1

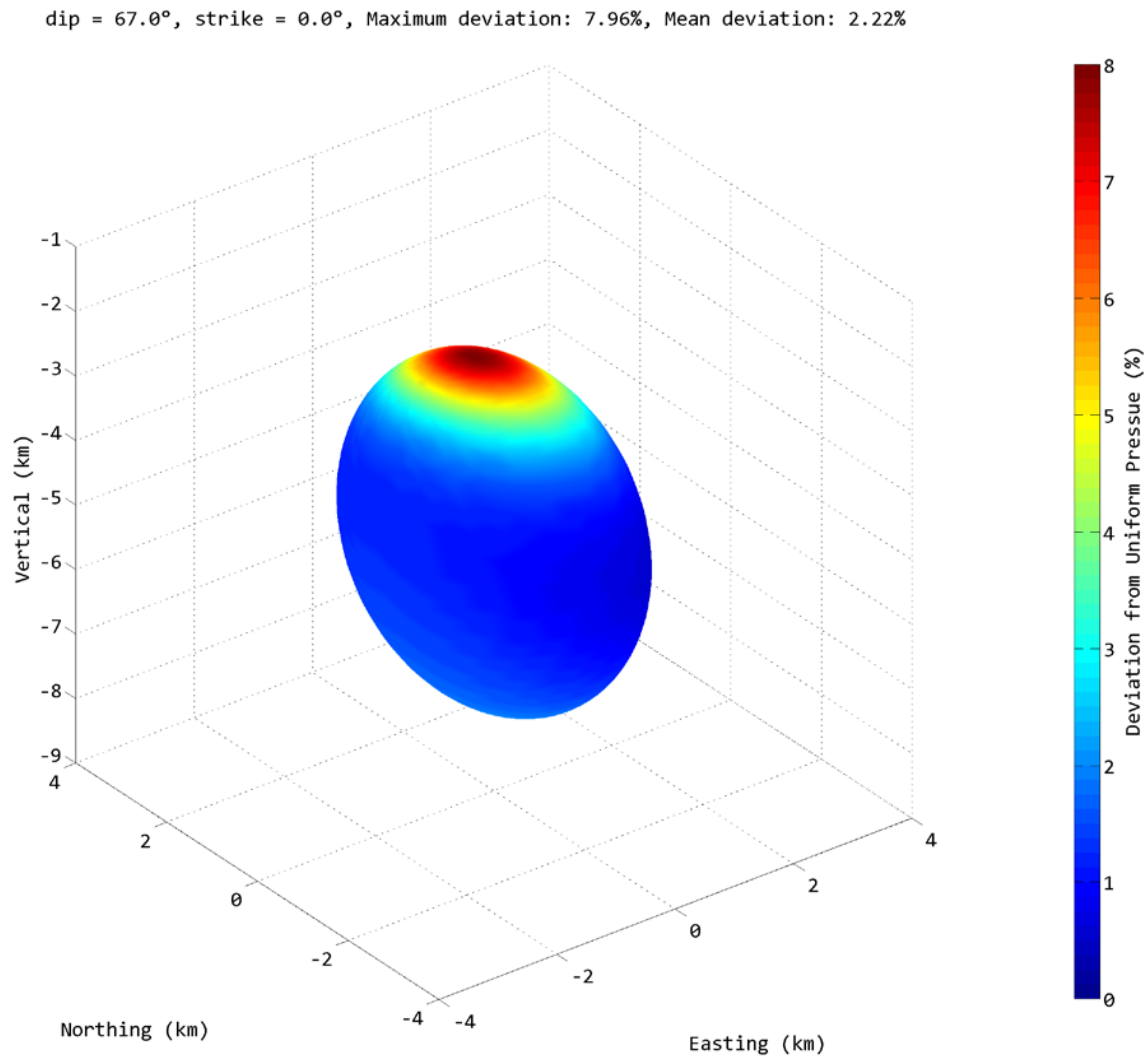


How Well is the Uniform Pressure Boundary Condition Satisfied?

Spheroid Parameters

a = 3000 m
b = 2000 m
Depth = 5000 m
P/μ = 1

As depth decreases the departure from uniformity increases.



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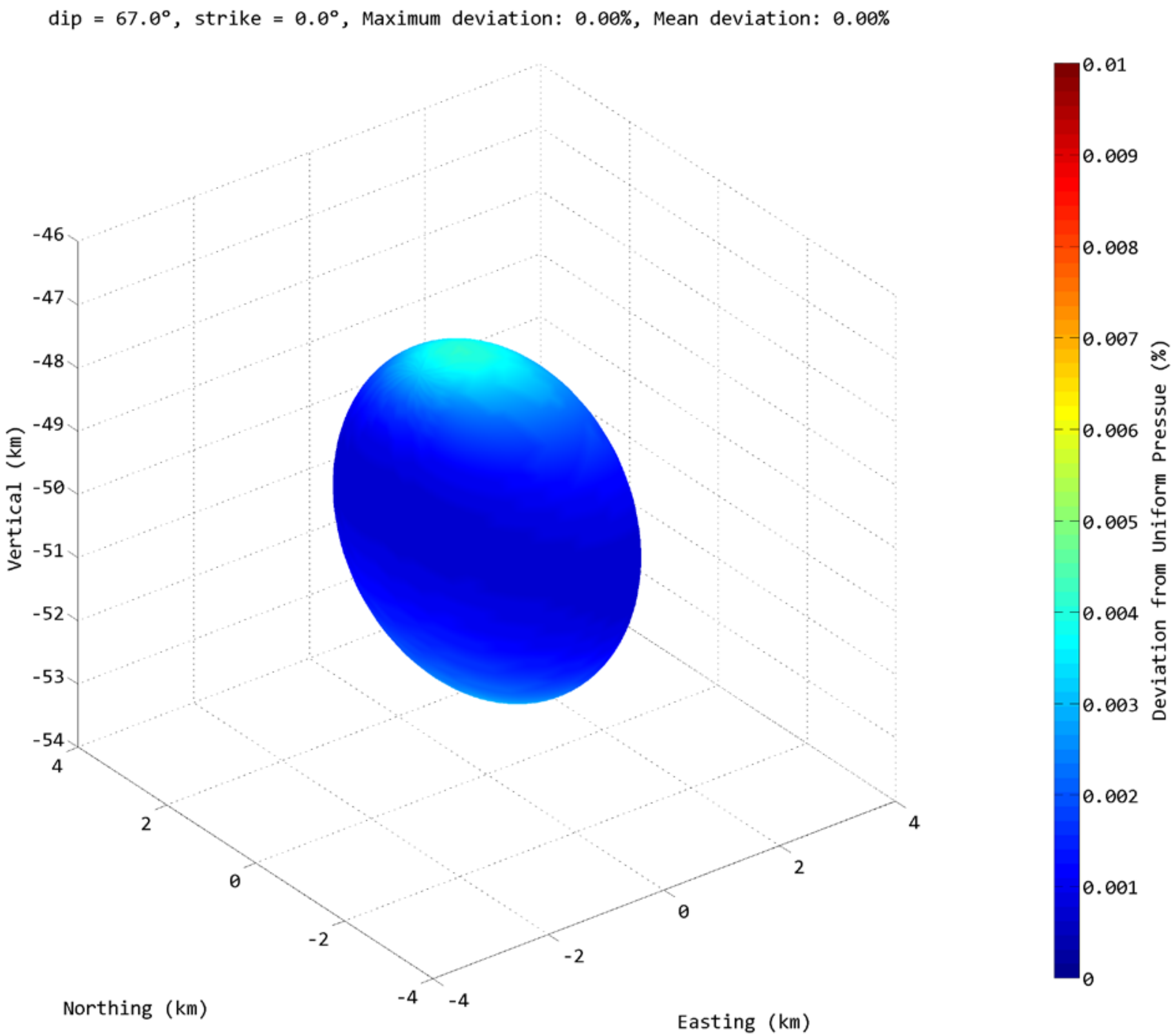
How Well is the Uniform Pressure Boundary Condition Satisfied?

Spheroid Parameters

a = 3000 m
b = 2000 m
Depth = 50000 m
P/μ = 1

As depth goes to infinity the deviation from uniformity goes to zero.



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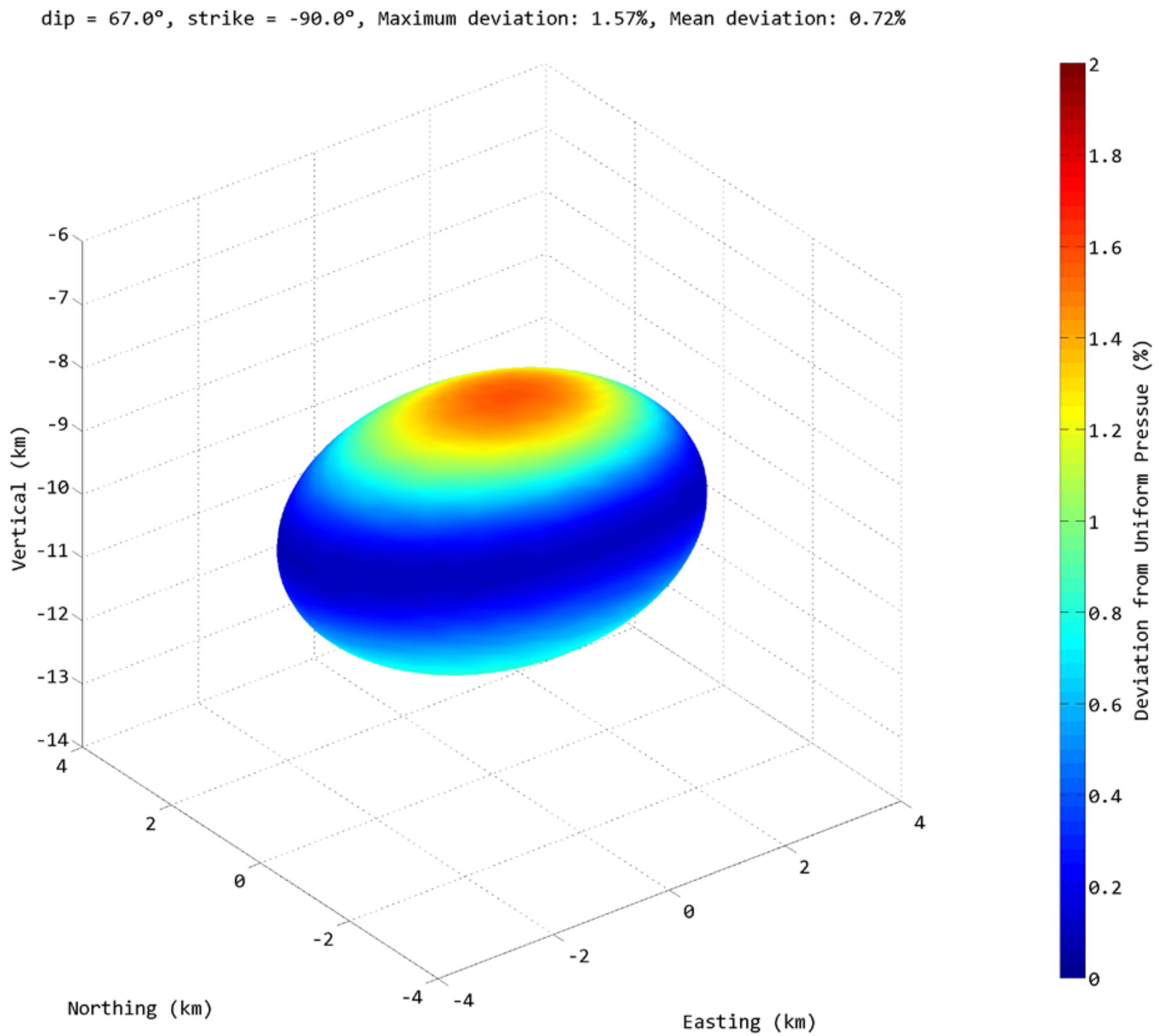
The Oblate Spheroid

Spheroid Parameters

a = 2000 m
b = 3000 m
Depth = 10000 m
P/μ = 1

Why does this work?
When $a < b$, then c becomes complex!



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When c is Complex, the Resultant Displacements are Still Real

Consider the full space version of Volterra's equation:

$$U_i = \int_{-c}^{+c} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \, \frac{\partial G_{i,3}}{\partial \xi_3} \right) d\xi$$

Re-write the equation for the oblate case:

$$U_i = \int_{-i\sqrt{b^2-a^2}}^{+i\sqrt{b^2-a^2}} \left(P_c \, \lambda \, \frac{\partial G_{i,j}}{\partial \xi_j} + P_d \, 2 \, \mu \, \frac{\partial G_{i,3}}{\partial \xi_3} \right) d\xi$$

We can then push the ***i*** into the variable of integration, ξ, and expand the result.

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Oblate Spheroid Deformation in a Full Space

Real Expressions

$$R = \left(4\,x3^2\,\xi^2 + \left(x1^2 + x2^2 + x3^2 - \xi^2\right)^2\right)^{1/4}$$

$$\theta = \mathrm{ArcTan}\left[a^2 - c^2,\ -2\,a\,c\right]$$

$$\phi = \frac{1}{2}\,\mathrm{ArcTan}\left[x1^2 + x2^2 + x3^2 - \xi^2,\ 2\,x3\,\xi\right]$$

$$\psi = \mathrm{ArcTan}\left[R\,\mathrm{Cos}[\phi] - x3,\ R\,\mathrm{Sin}[\phi] - \xi\right]$$

$$a1 = \frac{-2\,b^2\left(a^2 + 2\,b^2\right)c - 3\,a\,b^4\,\theta}{4\,a\,c^2\left(-b^2 + a^2\,(-1 + nu)\right) + 2\,b^2\,c\left(b^2 + 2\,a^2\,(-1 + nu)\right)\theta + a\,b^4\,(1 + nu)\,\theta^2}$$

$$b1 = \frac{4\,a^2\,c\,(-1 + nu) + 2\,b^2\,c\,(-3 + 2\,nu) + a\,b^2\,(-5 + 4\,nu)\,\theta}{4\,a\,c^2\left(-b^2 + a^2\,(-1 + nu)\right) + 2\,b^2\,c\left(b^2 + 2\,a^2\,(-1 + nu)\right)\theta + a\,b^4\,(1 + nu)\,\theta^2}$$

$$U1 = \frac{P\,x1}{4\,mu}\left(a1\left(\frac{\xi\,\mathrm{Cos}[\phi]}{R} - \frac{b^2}{(x3 - R\,\mathrm{Cos}[\phi])^2 + (\xi - R\,\mathrm{Sin}[\phi])^2}\left(\frac{\xi\,\mathrm{Cos}[\phi] + x3\,\mathrm{Sin}[\phi]}{R} - 2\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\phi]\right) + \psi\right) + b^2\,b1\left(\frac{2\,x3\,\xi - R\,(\xi\,\mathrm{Cos}[\phi] + x3\,\mathrm{Sin}[\phi])}{(x3 - R\,\mathrm{Cos}[\phi])^2 + (\xi - R\,\mathrm{Sin}[\phi])^2} - \psi\right)\right)$$

$$U2 = U1\,\frac{x2}{x1}$$

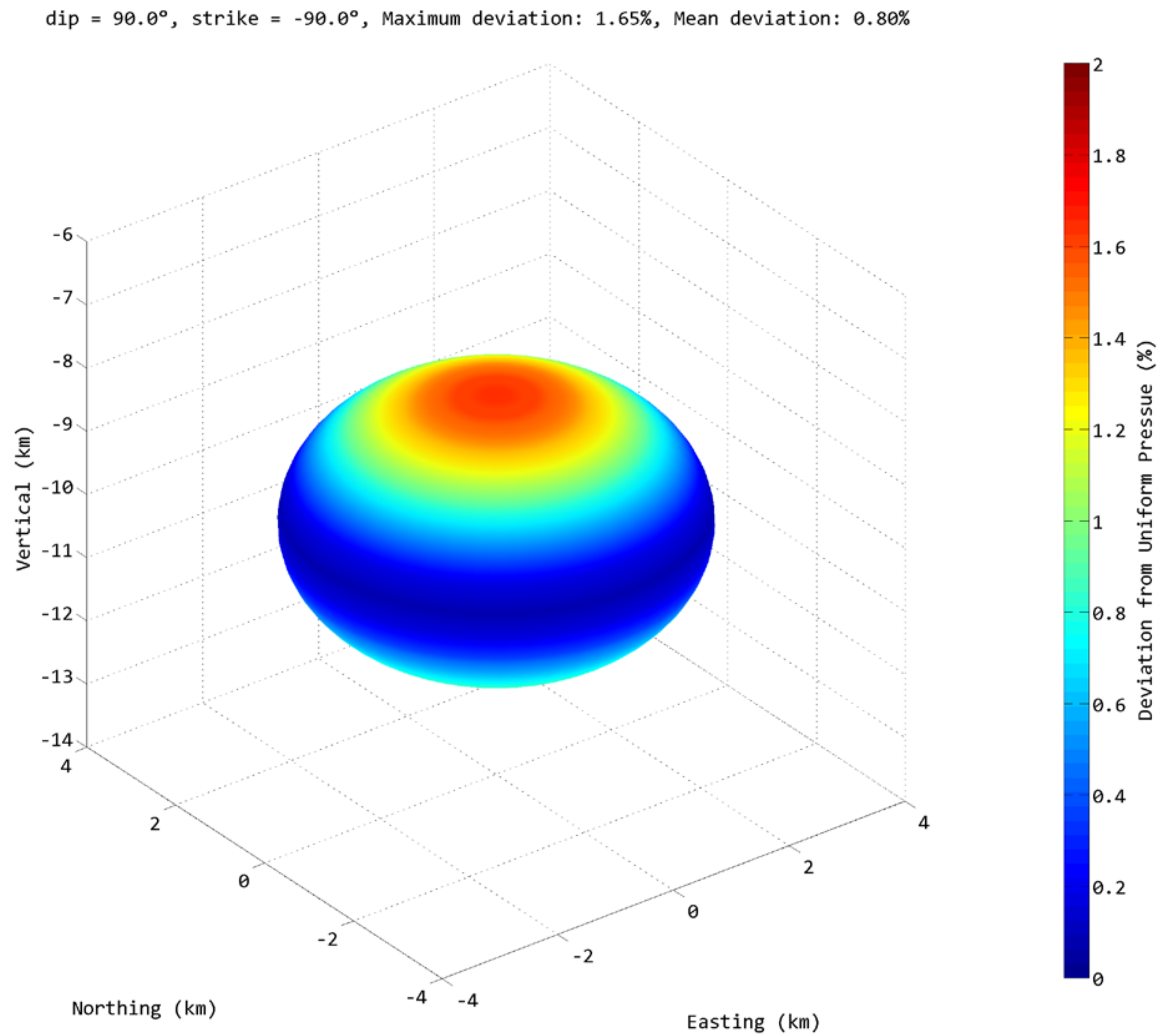
$$U3 = \frac{P}{4\,mu}\left(a1\left(4\,(-1 + nu)\,x3\,\psi + \frac{x3\,\xi\,\mathrm{Cos}[\phi] + \left(\xi^2 - b^2\right)\mathrm{Sin}[\phi]}{R} - (5 - 4\,nu)\,R\,\mathrm{Sin}[\phi]\right) + 2\,b^2\,b1\,(x3\,\psi + R\,\mathrm{Sin}[\phi])\right)$$

But, from a practical point of view, re-writing the equations is not necessary, provided your programming language supports complex numbers – the imaginary part of the output always cancels to zero (± rounding errors).

The Oblate Spheroid as a Goes to Zero

Spheroid Parameters

a = 2000 m
b = 3000 m
Depth = 10000 m
P/μ = 1



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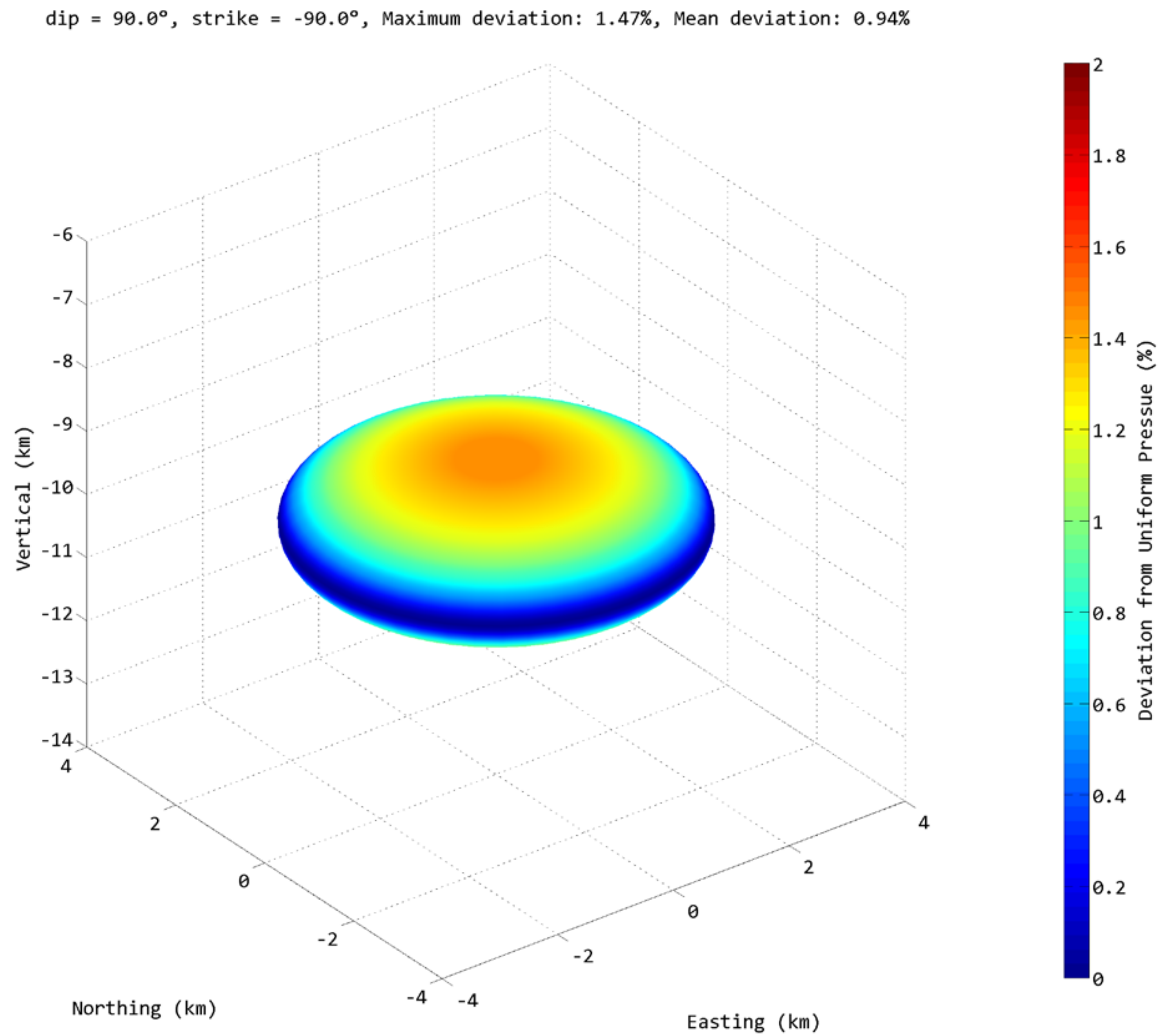
The Oblate Spheroid as a Goes to Zero

Spheroid Parameters

a = 1000 m
b = 3000 m
Depth = 10000 m
P/μ = 1



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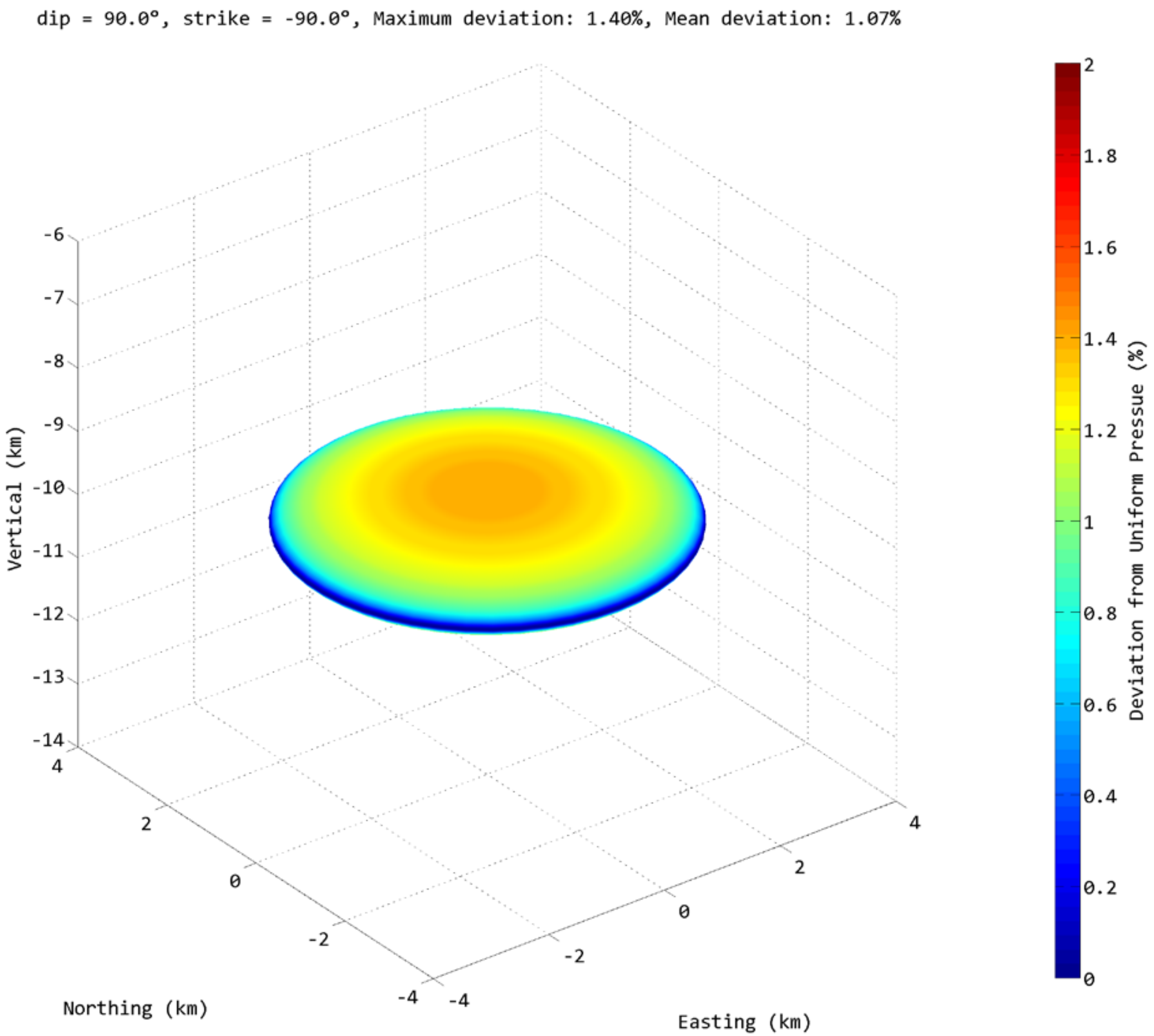
The Oblate Spheroid as a Goes to Zero

Spheroid Parameters

a = 500 m
b = 3000 m
Depth = 10000 m
P/μ = 1



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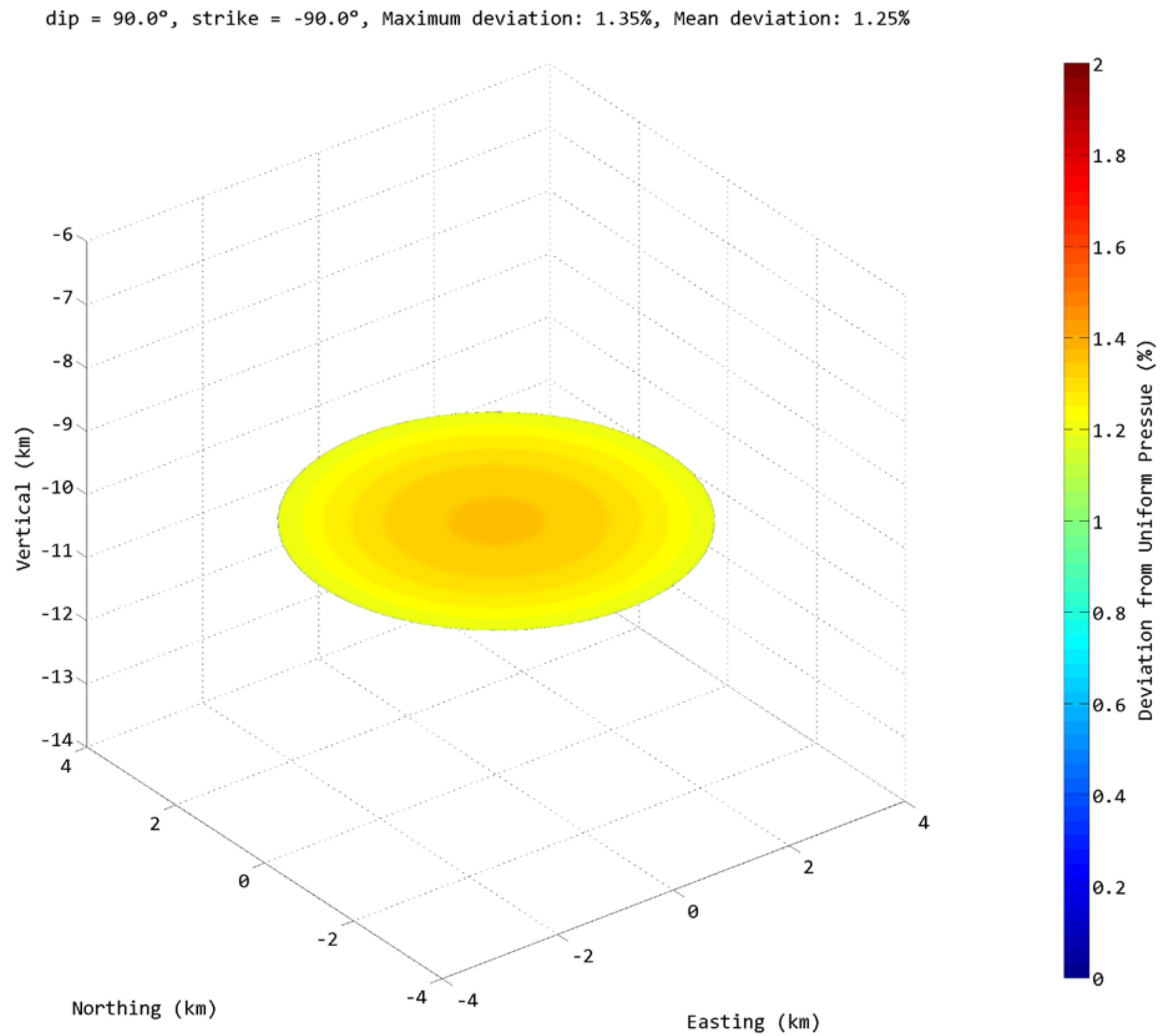
The Oblate Spheroid as a Goes to Zero

Spheroid Parameters

a = 0 m
b = 3000 m
Depth = 10000 m
P/μ = 1



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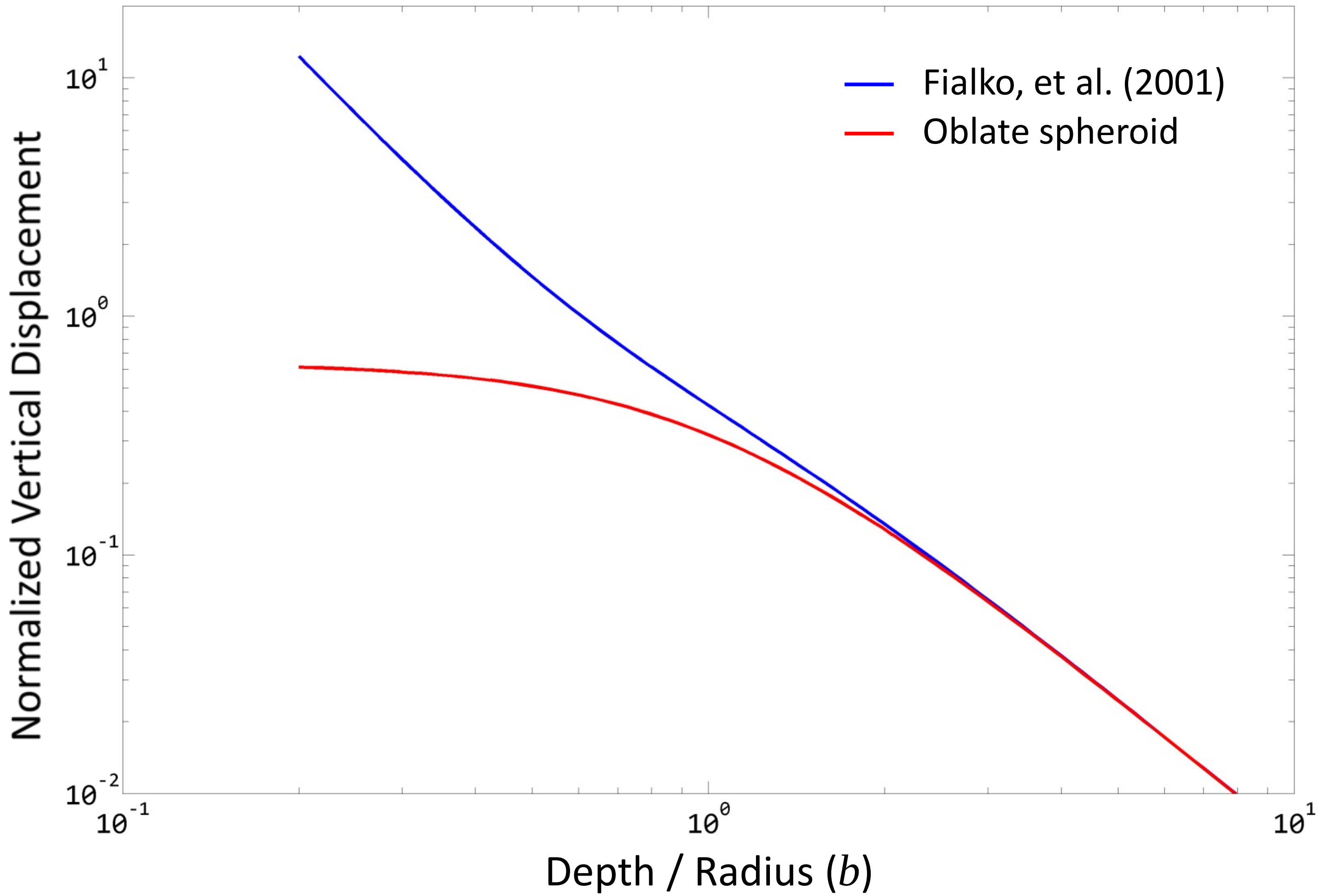
Comparison to Fialko's Penny-shaped Crack

Fialko's solution is exact

Oblate spheroid is approximate

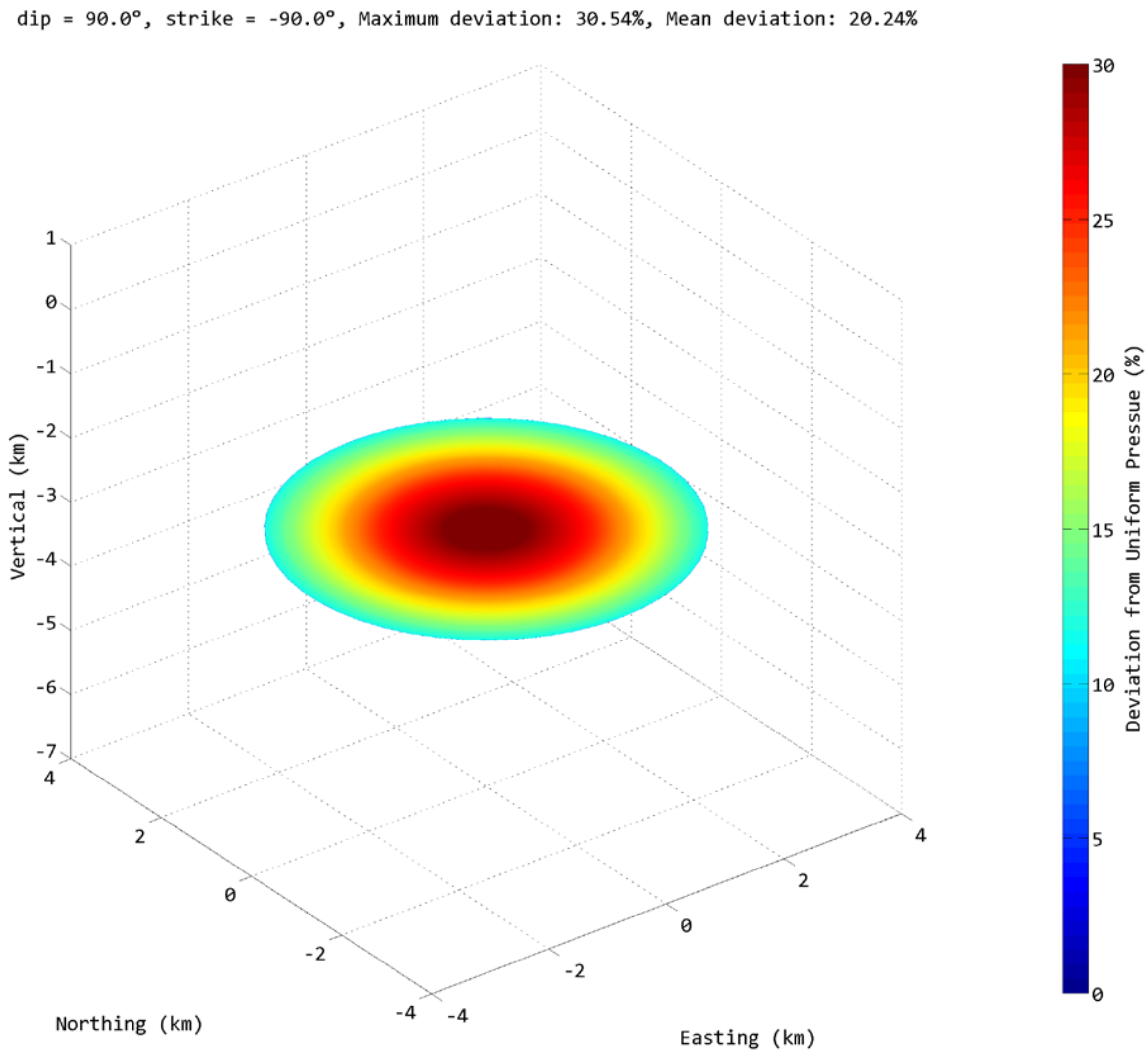


Oblate spheroid with a = 0 equivalent to the solution of Sun, 1969

The Oblate Spheroid as a Goes to Zero

Spheroid Parameters

a = 0 m
b = 3000 m
Depth = 3000 m
P/μ = 1



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Pressure / Volume Relationship

Volume Prior to Pressurization:

$$V_0 = \frac{4}{3}\,\pi\, a\, b^2$$

Volume After Pressurization:

$$V_1 = \frac{4}{3}\,\pi\,(a + \delta \mathrm{a})\,(b + \delta \mathrm{b})^2$$

Volume Change:

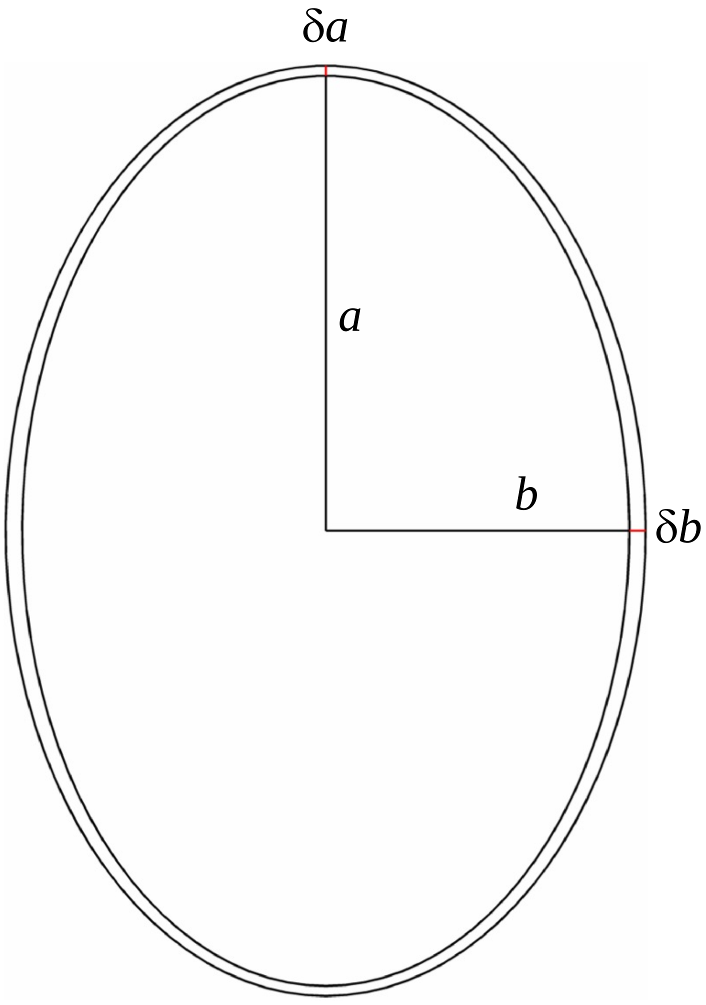
$$\Delta \mathrm{V} = V_1 - V_0$$

Partial derivatives with respect to axes:

$$\frac{\partial V}{\partial a} = \frac{4\,\pi\, b^2}{3} \quad \text{and} \quad \frac{\partial V}{\partial b} = \frac{8\,\pi\, a\, b}{3}$$

Approximate volume change:

$$\Delta \mathrm{V} \approx \frac{4}{3}\,\pi\, b\,(b\,\delta \mathrm{a} + 2\, a\,\delta \mathrm{b})$$



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Pressure / Volume Relationship

Derive analytical expressions for δa and δb in a full space:

$$\delta \mathrm{a} = -\frac{\Delta \mathrm{P}}{2\,\mu}\left(a_1\,c + \left(2\,c + a\log\left(\frac{a-c}{a+c}\right)\right)\left(2\,a_1\,(1-\nu) - b^2\,b_1\right)\right)$$

$$\delta \mathrm{b} = -\frac{\Delta \mathrm{P}}{4\,\mu}\,b\left(2\,a\,c\,b_1 + \left(-a_1 + b^2\,b_1\right)\log\left(\frac{a-c}{a+c}\right)\right)$$

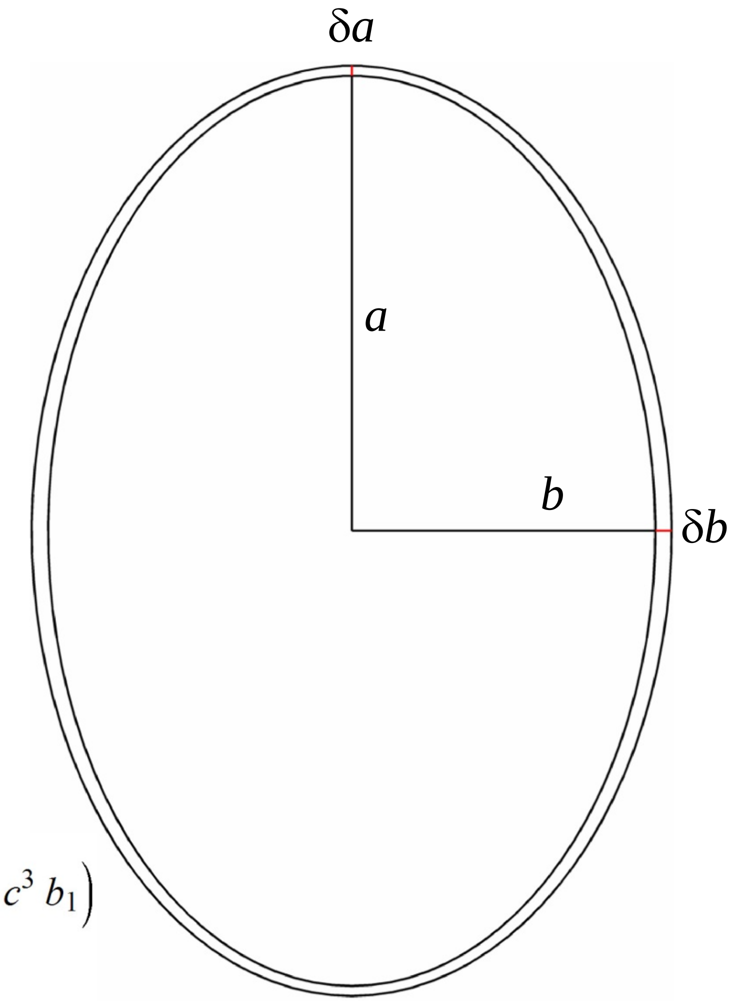
Insert into:

$$\Delta \mathrm{V} \approx \frac{4}{3}\,\pi\,b\,(b\,\delta \mathrm{a} + 2\,a\,\delta \mathrm{b})$$

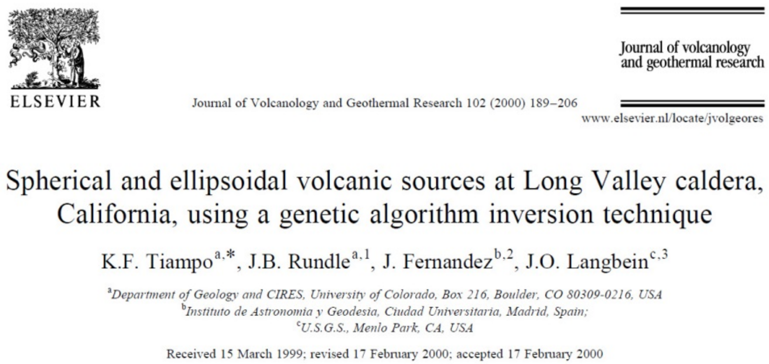
Giving the key relationships:

$$\Delta \mathrm{V} \approx \frac{2\,b^2\,\pi\,\Delta \mathrm{P}}{3\,\mu}\left(a_1\left(a\log\left(\frac{a-c}{a+c}\right)(-1+2\,\nu) + c\,(-5+4\,\nu)\right) - 2\,c^3\,b_1\right)$$

$$\Delta \mathrm{P} \approx \frac{3\,\Delta \mathrm{V}\,\mu}{2\,\pi\,b^2}\,\frac{1}{a_1\left(a\log\left(\frac{a-c}{a+c}\right)(-1+2\,\nu) + c\,(-5+4\,\nu)\right) - 2\,c^3\,b_1}$$



Comparison to Other Approximations



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Journal of Volcanology and Geothermal Research 102 (2000) 189–206

Journal of volcanology and geothermal research

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Spherical and ellipsoidal volcanic sources at Long Valley caldera, California, using a genetic algorithm inversion technique

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[b]Instituto de Astronomia y Geodesia, Ciudad Universitaria, Madrid, Spain;
[c]U.S.G.S., Menlo Park, CA, USA

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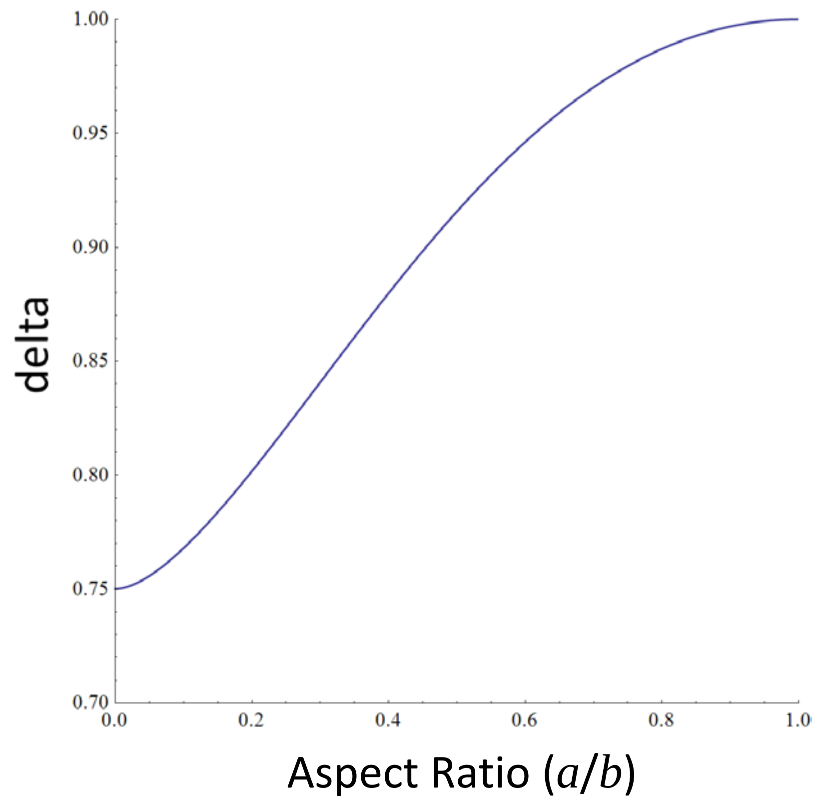
Tiampo et al., 2000, derived an approximation of volume change presented there in equation 18:

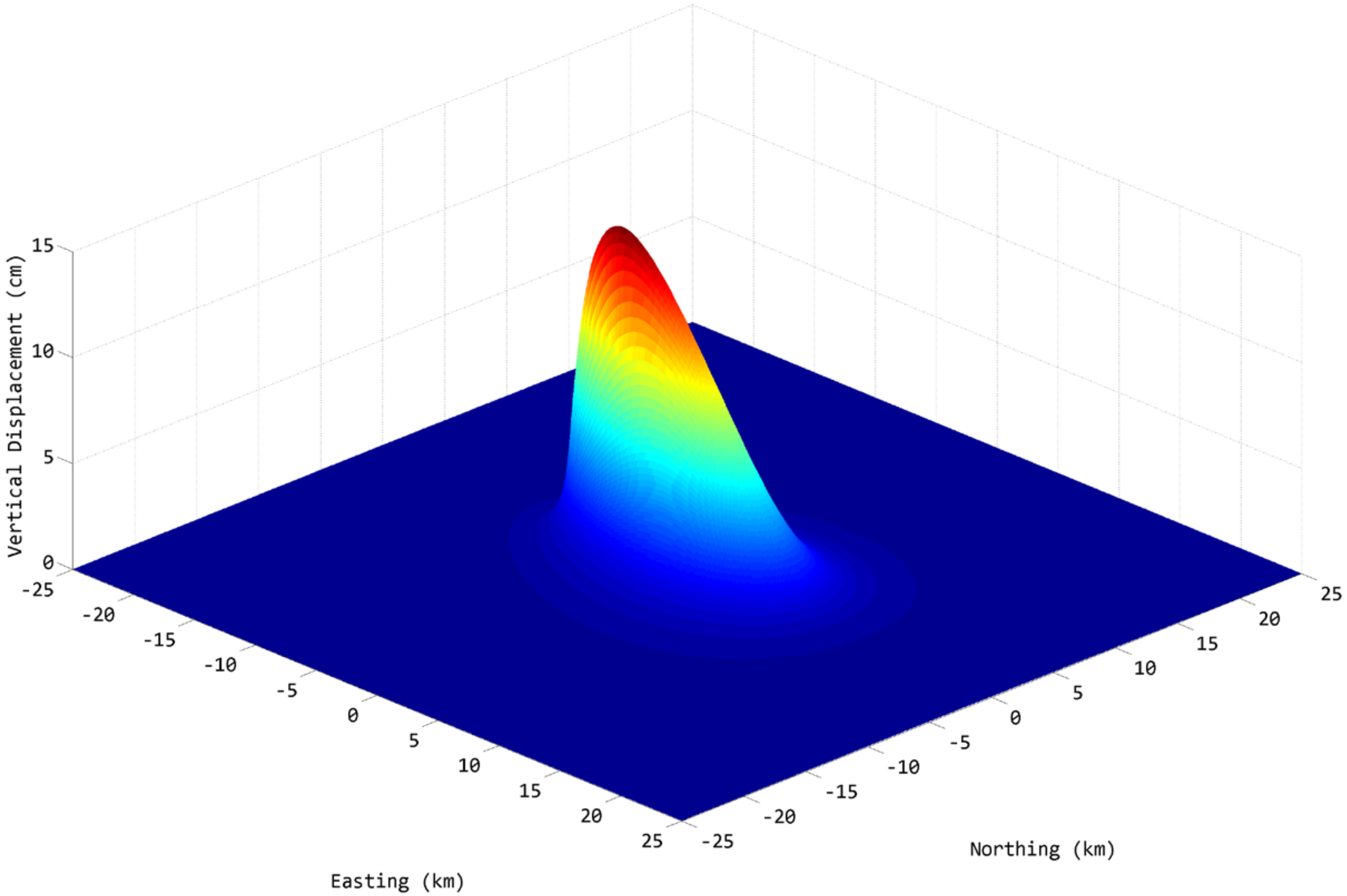
$$\Delta V \approx \frac{\Delta P \, b^2 \, \pi \, a}{\mu}$$

Compared to:

$$\Delta V \approx \frac{\Delta P \, b^2 \, \pi}{\mu} \, \frac{2}{3} \left(a_1 \left(a \log\left(\frac{a-c}{a+c}\right)(-1+2\,\nu) + c\,(-5+4\,\nu) \right) - 2\,c^3\, b_1 \right)$$



The End



Please visit:

volcanoes.usgs.gov/software/spheroid

for Matlab code and Mathematica notebooks

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